

# Planar Two-particle Coulomb Interaction: Classical and Quantum Aspects

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Abstract:

The classical and quantum aspects of planar Coulomb interactions have been studied in detail. In the classical scenario, Action Angle Variables are introduced to handle relativistic corrections, in the scheme of time-independent perturbation theory. Complications arising due to the logarithmic nature of the potential are pointed out.

In the quantum case, harmonic oscillator approximations are considered and effects of the perturbations on the excited (oscillator) states have been analysed.

In both the above cases, the known 3+1-dimensional analysis is carried through side by side, for a comparison with the 2+1-dimensional (planar) results.

## I. Introduction

In recent years, physics of the lower dimensional systems are playing more and more important roles, not only as toy models for the physical world, but also as an independent entity. On the one hand, effectively 1+1 and 2+1-dimensional systems have been constructed, while on the other hand, rich theoretical structures have been unearthed in these low dimensions.

In the present work, we have studied in some details, the classical and quantum aspects of planar Coulomb interactions. We have also attempted to estimate effects of relativistic corrections. The Darwin Lagrangian [1], (and the Hamiltonian), that we have considered has appeared before in [2]. Their roles have been further elaborated in [3], where the Coulomb interaction between anyons [4], has been considered. These studies are done in the spinning particle model of anyon, proposed by us [5]. Unlike the 3+1-dimensional Coulomb problem [6], the logarithmic nature of the Coulomb potential poses computational as well as conceptual problems. Instead of trying for a full solution of the problem, we obtain the (degenerate) frequency in the Action Angle Variable (AAV) formulation [6], for a restricted set of orbits. Then we apply the time independent perturbation theory [6] in the lowest order in  $\eta = v/c$ , ( $v$  and  $c$  being the velocities of the particle and light), for heavy and slowly moving particles, to study the relativistic corrections. Once again these steps are not straightforward and we have to make some ansatz which are justified numerically in a later chapter. We have shown that the perturbation lifts the degeneracy. However, the relations connecting the original coordinates to the AAVs are not well defined for the whole range of variables. Hopefully, this problem might turn out to be more of a computational nature.

In the quantum case, we have started by performing a harmonic oscillator approximation about the potential minimum and have compared the 3+1 and 2+1-dimensional results. Interestingly, the binding energy obtained from classical and semiclassical (or quantum) approaches are not widely different in the planar case, whereas they differ by orders of magnitude in the 3+1-dimensional case. The classical and quantum cases differ in the value of the angular momentum of the particle. In the former, we obtain it from the particular potential profile we have considered, whereas in the latter, we simply assume it to be of the order of  $\hbar$ . Next we study the effects of perturbation in the harmonic oscillator excited states [7], which contribute only from the second order in  $\eta$ . Here we face a conceptual problem concerning the range of the radial coordinate variable. The correction depends on the boundary condition, where we fix the zero of the logarithmic potential.

As the foreward indicates, it may seem that our analysis has both solved and unearthed problems, (at least as far as the study of perturbations are concerned), of equal number! This obviously underlines the fact that we must know the classical and quantum aspects of planar potential problems more thoroughly. Also it is worthwhile to mention that some of the troublesome questions are inherently present in 2+1-dimensional system, which are not a reduction of an original 3+1-dimensional system, (with a translation symmetry in one direction), as is normally encountered.

The paper is organised as follows. In section II, we reproduce the 3+1-dimensional ( $\frac{1}{r}$ ) Coulomb problem results in terms of AAV. This actually outlines the approach to be

pursued for the planar ( $\ln \frac{r}{r_0}$ ) Coulomb potential, which is carried out in section **III**. The validity of the ansatz, introduced in section **III**, are checked in section **IV**. Section **V** constitutes a brief analysis of the time independent perturbation analysis of the relativistic corrections of the present model, where some problems in the numerical study have been stressed. Section **V** comprises of the quantum analysis where again the logarithmic potential raises some non-trivial questions. The 3+1-dimensional analysis is also carried out side by side for completeness.

## II. Classical Analysis in 3+1-dimensions

In this section, we briefly discuss the text book problem [6] of 3+1-dimensional Coulomb problem in (AAV), to outline the approach and for comparison with the 2+1-dimensional case at various stages.

For the cases where explicit solutions of the time evolution of coordinates or the equations of orbits are not required, (or are not computable), time independent perturbation theory comes to the rescue. If one can express the original coordinates and momenta in terms of AAV, the perturbations in the frequencies (of periodic systems) are computable in a systematic way and with little fuss. However, the catch is in the former task! As is well known this poses no problem in 3+1- dimensions, but the analogous problem runs in to severe computational trouble in 2+1-dimensions owing to the logarithmic Coulomb potential.

The Lagrangian and Hamiltonian of two interacting particles of mass  $2m$  each and charge  $Q$  and  $-Q$  are,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\psi}^2) + \frac{Q^2}{4\pi\epsilon_0 r}, \quad (1)$$

$$H = \frac{p_r^2}{2m} + \left(\frac{l^2}{2mr^2} - \frac{Q^2}{4\pi\epsilon_0 r}\right) = KE + V_{eff}, \quad (2)$$

where the  $r$  and  $\psi$  are the relative coordinates in the plane of the motion and  $m$  is the reduced mass. The conjugate momenta are

$$p_r = m\dot{r}, \quad p_\psi = mr^2\dot{\psi} = l \equiv \text{constant}.$$

In Figure (1),  $\frac{4\pi\epsilon_0}{Q^2}V_{eff} = \left(\frac{A}{r^2} - \frac{1}{r}\right)$  vs.  $r$  is plotted, where  $A = \frac{2\pi\epsilon_0 l^2}{mQ^2} \text{ metre} = 0.01 \text{ metre}$ , for later comparison with 2+1-dimensional results.

The action variables in a general  $r, \theta, \phi$  coordinate system are

$$J_k = \oint p_k dq_k, \quad k = r, \theta, \phi, \quad (3)$$

the integral being a line integral over complete periods of orbits in  $(q_k, p_k)$  plane. The generic central force Hamiltonian is

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta}) + V(r), \quad (4)$$

where  $p_k = \frac{\partial L}{\partial \dot{q}_k}$ . The total angular momentum  $l$  of (2) is related to this notation by  $l^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}$ . Here one obtains,

$$J_\phi = 2\pi p_\phi, \quad J_\theta = 2\pi(l - p_\phi), \quad (5)$$

$$J_r = \oint p_r dr = 2 \int_{r_1}^{r_2} \left( 2mE + \frac{mQ^2}{2\pi\epsilon_0 r} - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2} \right)^{\frac{1}{2}} dr, \quad (6)$$

where,  $r_1$  and  $r_2$  being the two roots of the integrand, are the turning points of the classical motion and  $V(r)$  is the Coulomb potential.  $E$  is the total energy of the system. This integral can be done directly or via contour method, with the result,

$$J_r = -(J_\theta + J_\phi) + \frac{\pi Q^2}{4\pi\epsilon_0} \sqrt{\frac{2m}{-E}}. \quad (7)$$

Now (7) can be inverted to write  $E$  in terms of the action variables only,

$$E = -\frac{mQ^4}{8\epsilon_0^2 (J_r + J_\theta + J_\phi)^2}. \quad (8)$$

The degenerate frequency  $\nu_k = \frac{\partial E}{\partial J_k}$  is,

$$\nu_r = \nu_\theta = \nu_\phi = \frac{mQ^4}{4\epsilon_0^2 (J_r + J_\theta + J_\phi)^3}. \quad (9)$$

By performing the indefinite integrals in

$$W = \int^\phi p_\phi d\phi + \int^\theta p_\theta d\theta + \int^r p_r dr, \quad (10)$$

one can see that Hamilton's characteristic function is completely separable. The angle variables are defined by  $w_k = \frac{\partial W}{\partial J_k}$  and inverting these relations the original coordinates are expressible in terms of the AAVs. Note that all the integrals are trivial here. From here the road to time independent perturbation theory is clear [6]. The procedure will be elaborated in the main body of our work.

On the other hand, one can get the crucial result of (8) in an easier way, albeit in a more restricted setting, by the following argument. The energy of the *circular* orbit is,

$$E_c = H|_{p_r=0, r=r_c} = -\frac{mQ^4}{8\epsilon_0^2 (J_\theta + J_\phi)^2}, \quad (11)$$

where  $r_c$  is the radius of the orbit,  $r_c = 2A$ , obtained from the condition  $-\frac{dV_{eff}}{dr}|_{r=r_c} = 0$ . For circular orbit we have  $p_r = 0$  and hence  $J_r = 0$ .

Now if one is assured of the existence of closed orbits, (for example by invoking Bertrand's theorem [6]), the frequencies  $\nu_k$  must be the same or at most rational multiples of each other. Hence, at least for the *completely degenerate* case, for non-circular orbits,  $J_r$  must enter the energy expression in the same way as  $J_\theta$  and  $J_\phi$  in (11) and so one recovers the general result (8), by simply replacing  $(J_\theta + J_\phi)$  by  $J_r + J_\theta + J_\phi$  in (11). Obviously the former rigorous result is much stronger, showing that there are no non-degenerate orbits in the 3+1 dimensional Coulomb problem. We will try to apply the latter scheme in the 2+1-dimensional problem under study.

### III. Classical Analysis in 2+1-dimensions

Let us start with the Coulomb Lagrangian and Hamiltonian,

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{Q^2}{2\pi\epsilon_0} \ln \frac{r}{r_0}, \quad (12)$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{Q^2}{2\pi\epsilon_0} \left( \frac{l^2\pi\epsilon_0}{Q^2mr^2} + \ln \frac{r}{r_0} \right) = KE + V_{eff}, \quad (13)$$

with  $p_\theta = mr^2\dot{\theta} = l$ , a constant, and  $p_r = m\dot{r}$ . The Coulomb potential is consistent with the Gauss law,  $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ , where  $\rho$  is the charge density,  $\mathbf{E}$  the electric field and  $\epsilon_0$  (or "permittivity") is just a property of the surrounding vacuum, which is related to the other property  $\mu_0$ , ("permeability"), by the relation  $\epsilon_0\mu_0 = \frac{1}{c^2}$ . Note that now  $\epsilon_0$  is such that  $\frac{Q^2}{\epsilon_0}$  has dimension of energy. The Coulomb potential is attractive for  $r < r_0$ ,  $r_0$  being the distance scale where the potential vanishes.

In Figure (2), we have shown the potential profile of  $V_{eff} = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{B}{r^2} + \ln \frac{r}{r_0} \right)$  for  $B = \frac{l^2\pi\epsilon_0}{mQ^2}$ . The graph is obtained by plotting  $\frac{2\pi\epsilon_0}{Q^2}V_{eff}$  vs.  $r$ , with  $r_0 = 1$  and the value of the parameter  $B = 10^{-5} \text{ (metre)}^2$ . Note that  $B$  is structurally similar but dimensionally different from the analogous parameter  $A$  in 3+1-dimensions.

The action variables are,

$$J_\theta = \oint p_\theta d\theta = 2\pi l, \quad J_r = \oint p_r dr. \quad (14)$$

The straightforward computation of  $J_r$  as in (6) is not possible since the transcendental nature of the integrand forbids explicit evaluation of the roots (or turning points). A contour integration might help but we have not been able to perform it. Instead we take recourse to the qualitative analysis described at the end of section (II).

The energy and radius of the circular orbit are,

$$E_c = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{1}{2} + \ln \left[ \sqrt{\frac{\epsilon_0}{2\pi m Q r_0}} \frac{J_\theta}{Q r_0} \right] \right) = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{1}{2} + \ln \frac{r}{r_0} \right), \quad (15)$$

$$r_c = \sqrt{\frac{\epsilon_0}{2\pi m Q}} \frac{J_\theta}{Q} = \sqrt{2B}. \quad (16)$$

Bertrand's theorem [6] assure us that closed orbits are allowed. Then *at least for the degenerate orbits*, where  $\nu_r = \nu_\theta$ , the energy is,

$$E = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{1}{2} + \ln \left[ \sqrt{\frac{\epsilon_0}{2\pi m Q r_0}} \frac{J_\theta + J_r}{Q r_0} \right] \right). \quad (17)$$

By inverting this relation we get,

$$J_\theta + J_r = \frac{r_0}{\sigma} \exp \left( \frac{2\pi\epsilon_0 E}{Q^2} \right), \quad (18)$$

where  $\sigma = \sqrt{\frac{\epsilon\epsilon_0}{2\pi m Q^2}}$  and  $e$  is the exponential number. We will give numerical estimates of the action integral in (18) later to show the validity of the above ansatz. Hence the single non-trivial frequency is,

$$\nu_r = \nu_\theta = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{1}{J_r + J_\theta} \right) = \frac{Q^2}{2\pi\epsilon_0 J_2}, \quad (19)$$

where we have defined a new set of action variables,  $J_\theta \equiv J_1$  and  $J_r + J_\theta \equiv J_2$ . The above is the unperturbed degenerate frequency.

The next task is to obtain expressions for the coordinates  $r$  and  $\theta$  in terms of AAVs. For this we define the Hamilton's characteristic function,

$$\begin{aligned} W &= \int^\theta p_\theta d\theta + \int^r p_r dr \\ &= \frac{J_1 \theta}{2\pi} + \int^r (2mE - \frac{J_1^2}{4\pi^2 r^2} - \frac{Q^2 m}{\pi \epsilon_0} \ln \frac{r}{r_0})^{\frac{1}{2}} . \end{aligned} \quad (20)$$

Note that the arbitrary  $r_0$  cancels out in the right hand side, but we carry it to keep an account of the dimensions and finally take  $r_0 = 1\text{metre}$ .

The two angle variables are,

$$w_1 = \frac{\partial W}{\partial J_1} = \frac{\theta}{2\pi} - \frac{J_1}{2\pi^2} \int \frac{dr}{r^2} (2mE - \frac{J_1^2}{4\pi^2 r^2} - \frac{Q^2 m}{\pi \epsilon_0} \ln \frac{r}{r_0})^{-\frac{1}{2}} , \quad (21)$$

$$w_2 = \frac{\partial W}{\partial J_2} = \frac{Q^2 m}{\pi \epsilon_0 J_2} \int dr (2mE - \frac{J_1^2}{4\pi^2 r^2} - \frac{Q^2 m}{\pi \epsilon_0} \ln \frac{r}{r_0})^{-\frac{1}{2}} \quad (22)$$

Once again we are stuck with integrals having logarithm inside. Let us rewrite the dimensionless angle variables in the form,

$$w_1 = \frac{\theta}{2\pi} - \sigma J_1 \sqrt{\frac{\pi}{2e}} \int \frac{dr}{r^2} (\alpha - \ln \frac{r}{r_0} - \frac{B}{r^2})^{-\frac{1}{2}} , \quad (23)$$

$$w_2 = \frac{1}{2\pi} \frac{\sqrt{2e}}{\sigma J_2} \int dr (\alpha - \ln \frac{r}{r_0} - \frac{B}{r^2})^{-\frac{1}{2}} , \quad (24)$$

where

$$\alpha = \ln \frac{\sigma J_2}{r_0}, \quad B = \frac{\epsilon_0 J_1^2}{4\pi m Q^2} = \frac{(\sigma J_1)^2}{2e}.$$

We use the following arguments to make the substitution,

$$\alpha - \ln \frac{r}{r_0} - \frac{B}{r^2} \equiv (r - r_1)(r_2 - r) \frac{2e}{\sigma^2 J_2^2} n^2 , \quad (25)$$

where  $r_1$  and  $r_2$  are the smaller and larger roots and  $n$  is an integer, to be determined later. From the nature of the potential well in Figure (2), we know that for the energy range  $0 > E > E_c$ , there are *only two roots*, (or turning points), of the expression of  $p_r$  in (20). Since in the classical regime, we have to stay inside the turning points for physical motion, the replacement in (25) is justified. Later we will compare the two expressions in (25) numerically. The constant factors are picked in such a way that when (20) is inverted, the libration coordinate  $r$  becomes a periodic function of  $w_2$ . From (24) and (25), we find,

$$w_2 = \frac{1}{2\pi n} \int \frac{dr}{\sqrt{(r - r_1)(r_2 - r)}} . \quad (26)$$

Next we come to the roots  $r_1$  and  $r_2$ . Indeed, numerical values of these roots will not suffice, since we require analytic expressions in terms of AAVs.

Computing the larger root  $r_2$  is easier. in the limit  $r \rightarrow \infty$ , we get

$$r_2 = r_0 \exp(\alpha) = \sigma J_2 . \quad (27)$$

However, estimation of the smaller root  $r_1$  is tricky. We use the algebraic consistency of the above relations, (in the special case of the parameter value we have chosen from Figure (2)). From (18) and (25), we have,

$$\begin{aligned} J_r &= 2 \frac{\sqrt{e}}{\sqrt{2\pi}} \sqrt{\frac{2\pi m Q^2}{e \epsilon_0}} \int_{r_1}^{r_2} dr \left( \alpha - \ln \frac{r}{r_0} - \frac{B^{\frac{1}{2}}}{r^2} \right) \\ &= \frac{2ne}{\pi \sigma^2 J_2} \int_{r_1}^{r_2} dr \sqrt{(r - r_1)(r_2 - r)} = \frac{en}{4\sigma^2 J_2} (r_2 - r_1)^2 . \end{aligned} \quad (28)$$

Using (27) we simplify (28) to get,

$$J_r = \frac{en}{4} \left( J_2 + \frac{r_1^2}{\sigma^2 J_2} - \frac{2r_1}{\sigma} \right) = \frac{en}{4} \left( J_r + J_1 + \frac{r_1^2}{\sigma^2 J_2} - \frac{2r_1}{\sigma} \right) . \quad (29)$$

Hence comparing left and right hand sides, we must have,

$$\frac{en}{4} = 1, \quad r_1^2 - 2\sigma r_1 J_2 + \sigma^2 J_1 J_2 = 0 .$$

Solving for the nearest integer value of  $n$  and for  $r_1$  we get,

$$n \approx 2, \quad r_1 = \sigma J_2 \pm \sigma J_2 \left( 1 - \frac{J_1}{J_2} \right)^{\frac{1}{2}} . \quad (30)$$

When we give the numerical estimates in the next section, it will be seen that  $J_2 > J_1$ . Since  $\sigma J_2 = r_2 > r_1$ , we find

$$r_1 \approx \sigma \left( \frac{J_1}{2} + \frac{J_1^2}{8J_2} \right) . \quad (31)$$

Later we will exhibit values of the roots obtained numerically and obtained from our approximate analytical expressions, for a comparison.

Taking all this into account, we finally get the angle variable as, (with  $n = 2$ ),

$$w_2 = \frac{1}{4\pi} \sin^{-1} \frac{2r - (r_1 + r_2)}{r_2 - r_1} , \quad (32)$$

$$\begin{aligned} w_1 &= \frac{\theta}{2\pi} - \frac{\sigma^2 J_1 J_2}{2\sqrt{2}e} \int \frac{dr}{r^2 \sqrt{(r - r_1)(r_2 - r)}} \\ &= \frac{\theta}{2\pi} - \frac{\sigma^2 J_1 J_2}{2\sqrt{2}e} \frac{[-r_1 r_2 + (r_1 + r_2)r - r^2]^{\frac{1}{2}}}{rr_1 r_2} . \end{aligned} \quad (33)$$

By inverting these relations we derive the desired expressions, with  $4\pi w_2 = \phi$ ,

$$\begin{aligned} r &= \frac{1}{2} [(r_2 - r_1) \sin \phi + (r_1 + r_2)] \\ &= \frac{\sigma J_2}{2} \left[ \left( 1 - \frac{\rho}{2} \right) \sin \phi + \left( 1 + \frac{\rho}{2} \right) \right] , \end{aligned} \quad (34)$$

$$\frac{\theta}{2\pi} = w_1 + \frac{1}{6\sqrt{2}e} \sqrt{\frac{1 - \sin\phi}{(1 + \sin\phi)^3}} (10 + 8\sin\phi + 2\sin^3\phi - 3\alpha + \alpha\sin\phi) , \quad (35)$$

where  $\rho = \frac{J_1}{J_2}$  is a small parameter. Before applying these relations in perturbation theory, in the next section we will demonstrate the validity of our assumptions numerically.

#### IV. Validity of the Ansatz - A Numerical Study

In the previous section, two crucial assumptions were introduced, in a rather cavalier fashion:

(i) The energy expression of for a circular orbit in (16) was generalized to any degenerate closed orbit in (47).

(ii) The expression  $\alpha - \ln \frac{r}{r_0} - \frac{B}{r^2}$  was replaced by a quadratic polynomial in  $r$  in (25).

Below we give the numerical comparison by going through the following steps.

(A): From Figure (2), we obtain a reasonable value of the parameter

$$B = \frac{\epsilon_0 J_1^2}{4\pi m Q^2} = 10^{-5}, \quad r_c = \sqrt{2B} .$$

For this particular  $B$ ,

$$E_c = \frac{Q^2}{2\pi\epsilon_0} \left( \frac{1}{2} + \ln(\sqrt{2B}) \right) = \frac{Q^2}{2\pi\epsilon_0} y .$$

We have put  $r_0 = 1$  metre. Hence the range of  $E$ , such that only two turning points occur, is  $0 > E > E_c = \frac{Q^2}{2\pi\epsilon_0} (-4.96)$ . In Table (1) below, we choose some typical values for  $E$ , (where  $\frac{2\pi\epsilon_0 E}{Q^2} = y$ ), and the corresponding roots  $r_1$  and  $r_2$  from Figure (2).

$y$	$r_1$	$r_2$	$\frac{\sqrt{2e}}{\pi} I \mid_n$
-1	0.00133	0.3678	0.261
-1.5	0.0014	0.22308	0.154
-2	0.00149	0.13523	0.089
-2.5	0.00159	0.08196	0.051
-3	0.00172	0.04958	0.027
-3.5	0.0019	0.02985	0.013
-4	0.00216	0.01773	0.006
-4.5	0.00264	0.01006	0.001

(B): First of all, let us check the smallness of  $\rho = \frac{J_1}{J_2}$ . For the two extreme values of  $E = 0$  and  $E_c = \frac{Q^2}{2\pi\epsilon_0} \ln(\sqrt{2eB})$ ,  $\rho = \sqrt{2eB} \approx 0.0074$  and  $\rho_c = 1$ , (since for circular orbits,  $J_1 = J_2$ ). We have,

$$J_1 = \sqrt{\frac{4\pi B m Q^2}{\epsilon_0}} = \frac{\sqrt{2eB}}{\sigma} \equiv \frac{x}{\sigma} \approx \frac{0.0074}{\sigma} ,$$

$$J_2 = \sqrt{\frac{2\pi m Q^2}{e\epsilon_0}} \exp\left(\frac{2\pi\epsilon_0 E}{Q^2}\right) = \frac{1}{\sigma} \exp\left(\frac{2\pi\epsilon_0 E}{Q^2}\right) .$$

We now compare the above results with our analytical expressions for  $r_1$  and  $r_2$  in (31) and in (27). Let us take a generic value  $E^a$  and the roots  $r_{1,2}^a$ . Thus we have  $E^a = \frac{Q^2}{2\pi\epsilon_0}y$ . We also have

$$J_1^a = \frac{x}{\sigma}, \quad J_2^a = \frac{1}{\sigma} \exp\left(\frac{2\pi\epsilon_0 E^a}{Q^2}\right) = \frac{1}{\sigma} \exp(y) \quad \rho = x e^{-y}.$$

Hence the roots are,

$$r_1 = \frac{\sigma J_1}{2} + \frac{\sigma J_1}{8} \alpha = \frac{x}{2} + \frac{x^2 e^{-y}}{8},$$

$$r_2 = \sigma J_2 = \exp(y).$$

This are listed in Table (2) below.

$y$	$r_1 = \frac{x}{2}(1 + \frac{x e^{-y}}{4})$	$r_2 = e^y$	$\frac{e(r_2 - r_1)^2}{2\sigma r_2}$
-1	0.00371	0.36788	0.49
-1.5	0.00372	0.22313	0.293
-2	0.00374	0.13534	0.174
-2.5	0.00377	0.08208	0.102
-3	0.00382	0.04979	0.058
-3.5	0.00391	0.0302	0.033
-4	0.00406	0.01832	0.015
-4.5	0.0043	0.01111	0.006

(C): We here show the comparison between the two forms of  $J_r$ . One is obtained by the following numerical integration, with the limits (obtained numerically) in Table (1).

$$J_r|_n = 2\sqrt{\frac{Q^2 m}{\pi\epsilon_0}} \int_{r_1}^{r_2} dr \left( \frac{2\pi\epsilon_0 E}{Q^2} - \frac{\pi\epsilon_0 J_1^2}{4\pi^2 m Q^2 r^2} - \ln \frac{r}{r_0} \right)^{\frac{1}{2}}$$

$$= 2\sqrt{\frac{Q^2 m}{\pi\epsilon_0}} \int_{r_1}^{r_2} dr \left( \frac{2\pi\epsilon_0 E}{Q^2} - \frac{B}{r^2} - \ln \frac{r}{r_0} \right)^{\frac{1}{2}}$$

$$= 2\sqrt{\frac{Q^2 m}{\pi\epsilon_0}} I|_n = \frac{\sqrt{2e}}{\pi\sigma} I|_n. \quad (36)$$

On the other hand, we have the analytical result from (28),

$$J_r = \frac{en}{4\sigma^2 J_2} (r_2 - r_1)^2 = \frac{e(r_2 - r_1)^2}{2r_2 \sigma}. \quad (37)$$

This are displayed in Table 2, in order to compare between  $\frac{\sqrt{2e}}{\pi} I|_n$  and  $\frac{e(r_2 - r_1)^2}{2r_2}$  for different values of  $E$  and corresponding  $r_1$  and  $r_2$ , from the two Tables.

(D) In Figure (3), comparative studies of the substitution of the logarithmic function of  $r$  by the quadratic polynomial in  $r$ , in (25), is shown for different values of the energy. In fig.(3) we plot the two functions in (25) inside the range  $r_1^a$  to  $r_2^a$  for four values of  $E^a$ ,

$$f(r) \equiv \alpha - \ln \frac{r}{r_0} - \frac{B}{r^2} = \ln \frac{\sigma J_2}{r} - \left( \frac{\sigma J_1}{2er} \right)^2 = y - \ln(r) - \frac{x^2}{4e^2 r^2},$$

$$g(r) \equiv \frac{4e}{(\sigma J_2)^2} (r - r_1)(r_2 - r) = 4ee^{-2y} \left[ r - \frac{x}{2} \left( 1 + \frac{xe^{-y}}{4} \right) \right] [e^y - r].$$

Note that even though the natures of the curves are somewhat different, (the respective peaks being in different places), the results are reasonably satisfactory, since we only use the function inside the action integral and effects coming from the spatially separated maxima average out.

## V. Time Independent Perturbation Theory

In this section, we only reproduce the expressions required for first order perturbation of the frequency [6], where  $\eta = \frac{v}{c}$  is our small parameter. The perturbation term  $H_1$  in  $H = H_0 + \eta H_1$  is expressed in terms of the unperturbed AAVs,

$$\alpha_1(J_0) = H_1(\overline{w_0}, J_0), \quad (38)$$

where overbar denotes averaging over a complete period of  $w_0$ , with  $\nu_0 = \frac{\partial H_0}{\partial J_0}$ ,  $w_0 = \nu_0 t + B_0$ ,  $B_0$  being a time independent constant. After the averaging, it is legitimate to replace  $J_0$  by  $J$ , (which is correct to  $O(\eta)$ ), and one derives the frequency  $\nu$ , correct upto  $O(\eta)$ ,

$$\nu = \frac{\partial \alpha}{\partial J} = \nu_0 + \eta \frac{\partial \alpha_1}{\partial J}. \quad (39)$$

The perturbation terms that we consider has been discussed in detail in [2] and [3].

$$H = H_0 + \eta H_1 = \left[ \frac{p^2}{m} + \frac{Q^2}{2\pi\epsilon_0} \ln \frac{r}{r_0} \right] + \left[ a \left( 1 + \ln \frac{r}{r_0} \right) + \frac{j b}{m c r} \right], \quad (40)$$

where  $a = \frac{r_i p_i}{m c r}$  and  $b = \frac{\epsilon^{ij} r_i p_j}{m c r}$  are dimensionless variables and  $j$  is the spin of the free particle. In [3] it has been shown that in interacting systems of this form, there is a screening effect on the particle spin. The higher order terms [2] in  $a$  and  $b$  are dropped. We replace  $p_i$  by  $m v_i$  and note that both  $a$  and  $b$  are of  $O(\eta)$ . It should be kept in mind that the  $a$  dependent term (and higher order ones present in the original work [2]) are conventional ones but the  $j$  dependent  $b$  terms are the spin effects. Here our aim is to check the stability of the closed orbits once the perturbation is switched on.

It is immediately apparant from (34), (35) and (40) that the degeneracy in the frequency is lifted. But we also notice that due to the necessary averaging to be done with respect to the angle variable  $\phi = 4\pi w_2$ , the expression for  $\theta$  in (35) is not well defined, as the denominator may vanish for some value of the angle. It should be pointed out that the above mentioned problem crops up even in the  $j$  independent correction terms as well. Hence it is not clear whether the fault lies in the general perturbation scheme or in the several approximations that we have introduced.

## VI. Quantum Mechanics

We start this section by giving estimates of the binding energy of a quantum system in 3+1-dimensions. Following the analysis in [8], the total energy expression to be minimised

is

$$E(r = R) = \frac{\hbar^2}{2mR^2} - \frac{Q^2}{4\pi\epsilon_0 R}, \quad (41)$$

where  $R$  is the ground state "radius" of the atom.  $\hbar$  is a typical value of the angular momentum in the quantum domain. This gives us

$$R = \frac{4\pi\epsilon_0\hbar^2}{mQ^2} \equiv \frac{8\pi^2\hbar^2}{e}\bar{\sigma}^2,$$

$$E(R) = -\frac{mQ^4}{2.(4\pi\epsilon_0\hbar)^2} \equiv -\frac{Q^2e}{64\pi^3\epsilon_0\hbar^2}\frac{1}{\bar{\sigma}^2}.$$

This comes to roughly  $R \approx 0.52910^{-10}$  metre and  $E \approx -13.5$  eV. The exactness of the numerical value of  $E$  is merely a coincidence [8].

Following this approach in 2+1-dimensions, we minimise,

$$E(r = R) = \frac{\hbar^2}{2mR^2} + \frac{Q^2}{2\pi\epsilon_0} \ln \frac{R}{r_0}.$$

From this we derive,

$$R = \frac{2\pi\hbar}{\sqrt{e}}\sigma, \quad E = \frac{Q^2}{2\pi\epsilon_0} \ln \frac{2\pi\hbar\sigma}{r_0}. \quad (42)$$

The parameter  $\sigma$  is expressionwise identical in both cases but differ in dimension and is the introduced in section (III).

It is amusing to note the following point: From Figure (1) and Figure (2), we can get an order of magnitude idea of the angular momenta involved;

$$l = \frac{\sqrt{eA}}{2\pi\sigma} \approx 2.10^{-30} \text{ Joule} - \text{Second}, \quad 3 + 1 - \text{dim.},$$

$$l = \frac{e\sqrt{2B}}{2\pi\sigma} \cdot 2 + 1 - \text{dim.} \quad (43)$$

In estimating the numerical value of  $l$  in 3+1-dimension, the particle is taken to be an electron. Note that (at least in 3+1-dimension) these values of  $l$  are much larger than  $\hbar$ , the typical order of quantum angular momentum. Obviously this  $l$  value in 3+1 dimensions will drastically change the ground state energy from the correct one, making it much smaller. However, since the angular momentum occurs in the logarithm in the 2+1 dimensional energy expression, there is no order of magnitude change in the energy value.

One can do a harmonic oscillator approximation about the potential minima in Figures (1) and (2). We will follow this up in the 2+1 dimensional case with effects of the perturbations introduced in the previous section.

In 3+1 dimension, we expand the effective potential energy about  $r_c = \frac{\epsilon_0 J_1^2}{\pi m Q^2}$ ,

$$V_{eff} = \frac{Q^2}{4\pi\epsilon_0} \left( -\frac{1}{r_c + x} + \frac{\epsilon_0 J_1^2}{2\pi m Q^2} \frac{1}{(r_c + x)^2} \right), \quad (44)$$

and identify the quadratic  $x$ -term with the harmonic oscillator potential, and obtain the frequency ,

$$\Omega = \sqrt{\frac{Q^2}{4\pi\epsilon_0 m r_c^2}} = \frac{e\pi Q^2}{4\pi\epsilon_0 J_1^3 \bar{\sigma}^2} = \frac{\pi m Q^4}{2\epsilon_0^2 J_1^3} . \quad (45)$$

Hence the excitation levels above the ground state are

$$E_n = n\hbar\Omega , \quad n = 1, 2, \dots .$$

Identical analysis in 2+1 dimension leads to an expansion of  $V_{eff}$  about  $r_c = \sqrt{\frac{\epsilon_0}{\pi m}} \frac{J_1}{Q}$ ,

$$V_{eff} = \frac{Q^2}{2\pi\epsilon_0} \left( \ln \frac{r_c + x}{r_0} + \frac{\epsilon_0 J_1^2}{2\pi m Q^2 (r_c + x)^2} \right) , \quad (46)$$

which produces the frequency

$$\Omega = \frac{e}{2\sqrt{2}\pi m J_1 \sigma^2} = \frac{Q^2}{\sqrt{2}\epsilon_0 J_1} . \quad (47)$$

Note that  $\Omega$  is independent of the particle mass.

With the lowest order perturbation terms in (40),

$$H_1 = \frac{Q^2}{2\pi\epsilon_0} \eta \left[ \left( 1 + \ln \frac{r}{r_0} \right) \cos\theta + \frac{j}{mcr} \sin\theta \right] ,$$

where we have written down the vector products explicitly, we will now concentrate on the corrections in the energy, taking the harmonic oscillator excited states, (above the ground state), as the unperturbed ones. Using the notation of [7], let us consider only the ground state and the first excited state wave functions.

a) Ground state: non-degenerate,  $E = \hbar\Omega$ ,  $n = 0$ ,  $n_r = 0$ ,  $M = 0$ ,

$$u_{0,0}(r, \theta) = \sqrt{\frac{\lambda}{\pi}} \exp\left(-\frac{\lambda r^2}{2}\right) , \quad (48)$$

b) First excited state: doubly degenerate,  $E = 2\hbar\Omega$ ,  $n = 1$ ,  $n_r = 0$ ,  $M = \pm 1$ ,

$$u_{0,\pm}(r, \theta) = \frac{\lambda}{\sqrt{\pi}} r \exp\left(-\frac{\lambda r^2}{2}\right) \exp(\pm i\theta) . \quad (49)$$

Here  $\lambda = \frac{m\Omega}{\hbar}$  has dimension of  $(length)^2$  and  $n$ ,  $n_r$  and  $M$  are respectively the total, radial and orbital angular momentum quantum numbers, with  $M$  being an integer.

It is readily seen that  $H_1$  above is linear in  $\cos\theta$  and  $\sin\theta$  and hence *there are no first order effects in any of the states*.

Coming now to second order perturbation theory, we have

$$\Delta_2 E_m = \sum_{m'} \frac{|\langle m' | V | m \rangle|^2}{E_m - E_{m'}} . \quad (50)$$

For the ground state this reduces to

$$\Delta_2 E_0 = \frac{|\langle 1 | V | 0 \rangle|^2}{E_0 - E_1} + \frac{|\langle -1 | V | 0 \rangle|^2}{E_0 - E_{-1}}$$

$$= \frac{2}{E_0 - E_1} \left( \frac{Q^2 \eta}{4\pi \epsilon_0} \right)^2 \left[ \mathcal{A}^2 + \frac{j^2}{m^2 c^2} \mathcal{B}^2 \right]. \quad (51)$$

Here we have used the following relations,

$$\begin{aligned} \langle m' | \cos\theta / \sin\theta | 0 \rangle &\equiv \delta_{m',1} \pm \delta_{m',-1}, \\ \langle \pm 1 | \cos\theta | 0 \rangle &= \frac{1}{2\pi} \int e^{-i(\pm\theta)} \cos\theta d\theta = \frac{1}{2}, \\ \langle \pm 1 | \sin\theta | 0 \rangle &= \pm \frac{i}{2}. \end{aligned}$$

$\mathcal{A}$ , (a dimensionless quantity), and  $\mathcal{B}$ , (a quantity of dimension  $(length)^{-1}$ ), are defined as,

$$\begin{aligned} \mathcal{A} &= \langle 1 | \left( 1 + \ln \frac{r}{r_0} \right) | 0 \rangle = 2(\lambda)^{\frac{3}{2}} \int_0^R r^2 \left( 1 + \ln \frac{r}{r_0} \right) e^{-\lambda r^2} dr, \\ \mathcal{B} &= \langle 1 | \left( \frac{1}{r} \right) | 0 \rangle = 2(\lambda)^{\frac{3}{2}} \int_0^R r e^{-\lambda r^2} dr. \end{aligned} \quad (52)$$

We are coming to the value of  $R$ , the upper limit of integration. One can rewrite the result in (51) as,

$$\Delta_2 E_0 = -\frac{2}{\hbar \Omega} \left( \frac{Q^2 \eta}{2\pi \epsilon_0} \right)^2 (\lambda r_0^2)^3 (I_1^2 + \left( \frac{j}{m c r_0} \right)^2 I_2^2), \quad (53)$$

where,

$$I_1 = \int_0^{\frac{R}{r_0}} k^2 (1 + \ln k) \exp(-\lambda r_0^2 k^2) dk, \quad (54)$$

$$I_2 = \int_0^{\frac{R}{r_0}} k e(-\lambda r_0^2 k^2) dk = \frac{1}{2\lambda r_0^2} (1 - \exp^{-\frac{R}{r_0}}). \quad (55)$$

The indefinite integral with the logarithm in (54) is not obtainable in a closed form. We have the following identities [9],

$$\int_0^{\frac{R}{r_0}} k^2 \exp(-\lambda r_0^2 k^2) dk = \frac{1}{2} (\lambda r_0^2)^{-\frac{3}{2}} \gamma\left(\frac{3}{2}, \lambda R^2\right), \quad (56)$$

$$\int_0^\infty k^2 \exp(-\lambda r_0^2 k^2) dk = \frac{1}{2} (\lambda r_0^2)^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right), \quad (57)$$

$$\int_0^\infty k^2 \exp(-\lambda r_0^2 k^2) \ln(k) dk = \frac{\sqrt{\pi}}{8} (\lambda r_0^2)^{-\frac{3}{2}} (2 - \ln(4\mu) - C), \quad (58)$$

where  $\gamma$  and  $\Gamma$  are respectively the incomplete Gamma function and Gamma function and  $C$  is the Euler number. Note that in the limit  $R \rightarrow \infty$ ,  $\Delta_2 E_0$  in (53) becomes totally independent of  $r_0$ . However, for any finite value of  $R$ ,  $r_0$  survives in the  $\gamma$  function and in the integral in (58) as well.

The non-trivial issue is the  $r_0$  dependence of the final result, which is coming from the upper limit  $\frac{R}{r_0}$ . If  $R$  is infinity, then the  $r_0$ -dependence disappears but conceptually the behaviour of the particles is difficult to visualise: The attraction (repulsion) between opposite (similar) charges changes to repulsion (attraction), once the  $r_0$ -barrier is crossed. It might seem natural to consider the upper bound of  $r$  to be  $r_0$ , but this compactification is also not satisfactory.

Generally one encounters logarithmic potentials in physical (3 dimensional) space having total infinite amount of charge with a translation symmetry in one space direction and the questions addressed here are not always pertinent. But if one is truly in a planar world, with logarithmic interparticle potential, then these questions can not be avoided.

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Figure 3

The functions,

$$f(r) = y - \ln(r) - \frac{x^2}{4e^2r^2}, \quad g(r) = 4ee^{-2y}\left[r - \frac{x}{2}\left(1 + \frac{xe^{-y}}{4}\right)\right][e^y - r]$$

are plotted for  $y = -1, -2, -3, -4$ .

$$(3a) \quad y = -1$$

$$(3b) \quad y = -2$$

$$(3c) \quad y = -3$$

$$(3d) \quad y = -4$$

Figure 1  
Potential profile in 3+1-dimensions,  $A = 0.01$ .

Figure 2  
Potential profile in 2+1-dimensions,  $B = 10^{-5}$ .